ISRAEL JOURNAL OF MATHEMATICS, Vol. 36, Nos. 3-4, 1980

A COUNTER-EXAMPLE IN THE PARTITION CALCULUS FOR AN UNCOUNTABLE ORDINAL

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ABSTRACT

The main theorem of the paper is a counter-example in the partition calculus introduced by P. Erdös and R. Rado: If κ is a regular cardinal and $\alpha = \kappa^{\omega+1}$, then $\alpha \not\rightarrow (\alpha, 3)^2$. The proof is combinatorial. Other counter-examples are produced from this one through the pinning relation which was introduced by E. Specker.

§1. Introduction and terminology

P. Erdös and R. Rado [4] first introduced the ordinal relation $\alpha \rightarrow (\alpha, 3)^2$ in a paper in 1956 in which they generalize Ramsey's Theorem in many different ways. (For a definition of this relation, see the end of the section.) The main theorem of this paper is a counter-example in this partition calculus: If κ is a regular cardinal and $\alpha = \kappa^{\omega+1}$, then $\alpha \not\rightarrow (\alpha, 3)^2$.

In section 3, the pinning relation introduced by E. Specker [14] is used to apply the main theorem to other ordinals. In this way J. Baumgartner [1] has shown that the theorem can be extended to singular cardinals of uncountable cofinality. More generally, he has shown that if κ is a cardinal of uncountable cofinality, α is an ordinal with $\kappa^\omega < \alpha < \kappa^+$, and the cofinality of α is the same as the cofinality of κ, then α satisfies the negative partition relation $\alpha \not\rightarrow (\alpha, 3)^2$. I use his approach to show that the negative partition relation also holds for any ordinal $\alpha = \kappa^a$ which is the ordinal power of a cardinal κ of uncountable cofinality for some infinite decomposable ordinal a.

In section 4, I characterize for a regular uncountable cardinal κ, the ordinals of

[†]Part of this paper is a reworking of material that appeared in the author's Ph.D. thesis written under J. Baumgartner.

The preparation of this paper was partially supported by NSF grant MCS-7903384.

Received August 2, 1979 and in revised form March 12, 1980

cardinality κ which can be pinned to $\kappa^{\omega+1}$. This characterization shows the limits of the set of ordinals of cardinality κ to which the basic counter-example may be applied.

In section 5, in order to fit the results of the paper into what is already known, I discuss the partition relation $\alpha \rightarrow (\alpha, 3)^2$ in the case of α an ordinal power of a cardinal. I close with some open questions.

The rest of this section is devoted to a review of terminology and basic facts used in the study of the partition calculus.

The set theoretic usage is fairly standard. Each ordinal is identified with the set of its predessors. Since the axiom of choice is assumed throughout, cardinals are identified with initial ordinals. Denote by ω_α the initial ordinal occupying position α in the sequence of initial ordinals; and set $\omega_0 = \omega$. If κ is a cardinal, then κ^+ is the next largest cardinal.

The exponentiation in expressions of the form 2^κ is cardinal exponentiation. All other exponentiation is ordinal exponentiation.

If X is a set, then $|X|$ is the cardinality of X and $[X]^n$ is the set of n-element subsets of X. If X is a well-ordered set and α an ordinal, then $\operatorname{tp} X = \alpha$ means X is order-isomorphic to the ordinal α ordered by the membership relation. In this case, X has order type α.

If α is an ordinal, then $\operatorname{cf} \alpha$ denotes the least ordinal β which can be mapped onto a cofinal subset of α. A cardinal κ is regular if $\operatorname{cf} \kappa = \kappa$. Otherwise a cardinal is singular. The cofinality of an ordinal is always a regular cardinal. An ordinal α is decomposable if it can be represented as a sum $\beta + \gamma$ where $\beta, \gamma < \alpha$. Otherwise it is called indecomposable. It is well-known [12] that if α is indecomposable and $\alpha = B \cup C$, then either B or C has order type α. The indecomposable ordinals are those of the form ω^a where a is an ordinal.

Ramsey's theorem has been generalized in many ways. The relation $\alpha \rightarrow (\beta)^r_\gamma$ holds if and only if for every set A of type α and every function $R : [A]^r \rightarrow \gamma$, there are $\delta < \gamma$ and $B \subset A$ with $\operatorname{tp} B = \beta$ and $R[B]^r = \{\delta\}$. In this notation Ramsey's theorem is $\omega \rightarrow (\omega)^r_n$ for every $r, n < \omega$. The relation $\alpha \rightarrow (\beta_0, \beta_1, \cdots, \beta_{n-1})^r$ holds if and only if for every set A of type α and every function $R : [A]^r \rightarrow n$, there are $i < n$ and $B \subset A$ so that $\operatorname{tp} B = \beta_i$ and $R[B]^r = \{i\}$. The relation considered in the paper is a special case of the last. The relation $\kappa \rightarrow (\kappa)^{<\omega}_2$ holds if and only if for every set A of type κ and every function f from $[A]^{<\omega} = \bigcup_{n<\omega} [A]^n$ into 2, there is a set $B \subset A$ of type κ so that for each $n < \omega$, f is constant on $[B]^n$. If $\kappa \rightarrow (\kappa)^{<\omega}_2$, then κ is called a Ramsey cardinal.

Pinning is defined in section 3.

§2. The counter-example

THEOREM 2.1. *If κ is a regular cardinal, then $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$.*

If κ is countable, then $\kappa^{\omega+1}$ is $\omega^{\omega+1}$, which is known to have the negative partition relation. This result follows from work of F. Galvin [5] which shows that if α is ω^a where a is a countable decomposable ordinal, then α has the negative partition relation. So for the rest of this section assume κ is uncountable.

For each $n < \omega$, let $K(n)$ be the set of increasing sequences of ordinals less than κ of length n. Let $K = \bigcup_{n\in\omega} K(n)$. Let $<_l$ be the lexicographic ordering. Then for each $n < \omega$, $(K(n), <_l)$ has order type κ^n. Define another order $<$ on K by setting

$$a = (a_0, a_1, \cdots, a_{n-1}) < b = (b_0, b_1, \cdots, b_{m-1})$$

if and only if one of the following three conditions holds:

(i) $a_0 < b_0$,

(ii) $a_0 = b_0$ and $n < m$,

(iii) $a_0 = b_0$, $n = m$ and $a <_l b$.

Then $(K, <)$ has order type $\kappa^\omega \cdot \kappa = \kappa^{\omega+1}$.

Define $R : [K]^2 \to \{0,1\}$ by setting $R\{a,b\} = 1$ if and only if $a = (a_0, a_1, \cdots, a_{n-1})$ and $b = (b_0, b_1, \cdots, b_{m-1})$ where $n = m$ and there is some j with $1 < j < n$ so that

$$a_{j-1} < b_0 \leqq a_j < b_{j-1} \leqq a_{n-1} < b_j.$$

Notice that the value of R depends on the length of the sequences and their relative ordering but not on the particular elements of the sequences. So the ordinals less than κ form a set of indiscernibles for the coloring. E. Specker's coloring for ω^3 has an analogous property.

Lemmas 2.2 and 2.7 below complete the proof of Theorem 2.1.

LEMMA 2.2. *There is no three element set $Y \subseteq K$ with $R[Y]^2 = \{1\}$.*

PROOF. Assume, by way of contradiction, that $Y = \{a, b, c\}$ is a set of three elements with $R[Y]^2 = \{1\}$. Then a, b, c all have the same length n, and no two have the same first element. Write $a = (a_0, a_1, \cdots, a_{n-1})$, $b = (b_0, b_1, \cdots, b_{n-1})$ and $c = (c_0, c_1, \cdots, c_{n-1})$. Without loss of generality, we may assume $a_0 < b_0 < c_0$. From the definition of R, there are natural numbers j, k, and m so that

(i) $a_{j-1} < b_0 \leqq a_j < b_{j-1} \leqq a_{n-1} < b_j$,

(ii) $a_{k-1} < c_0 \leqq a_k < c_{k-1} \leqq a_{n-1} < c_k$,

(iii) $b_{m-1} < c_0 \leqq b_m < c_{m-1} \leqq b_{n-1} < c_m$.

It follows that $a_{n-1} < b_{n-1} < c_{n-1}$. Now $a_{j-1} < b_0 < c_0 < a_k$, so $j \leqq k$. Since $c_{k-1} \leqq a_{n-1} < b_{n-1} < c_m$, also $k \leqq m$. Furthermore, $b_{m-1} < c_0 < c_{k-1} \leqq a_{n-1} < b_j$, so $m \leqq j$. Thus $j = k = m$. Therefore $a_j < b_{j-1} = b_{m-1} < c_0 \leqq a_j$, which yields the contradiction $a_j < a_j$. So the lemma follows.

Some auxiliary definitions and lemmas are needed to prove there is no subset X of K of order type $\kappa^{\omega+1}$ with $R[X]^2 = \{0\}$.

The following lemma is well-known. One can prove it by induction on the natural numbers using the definition of regularity.

Lemma 2.3. *If μ is a regular cardinal and ν is a cardinal with $\nu < \mu$ and $n < \omega$, then $\mu^n \rightarrow (\mu^n)^1_\nu$.*

Definition 2.4. Suppose m and n are natural numbers and $m \leqq n$. If y is a sequence of length n, then $y \mid m$ is the initial segment of y of length m. (So $y \mid 0$ is the empty sequence.) If Y is a set of sequences of length n, then $Y \mid m = \{y \mid m : y \in Y\}$.

Definition 2.5. Suppose κ is a regular cardinal, $n < \omega$, $I \subset n+1$, $Y \subset K(n)$. Then Y is an *I-product set* if and only if $n \in I$ and for all $i \in I - \{n\}$, for all $y \in Y \mid i$, there are κ sequences z in $Y \mid (i+1)$ with $z \mid i = y$.

If Y is an I-product set where $Y \subset K(n)$ and $|I| = m+1$, then the elements y of Y can be written as concatenation of sequences, $y = s_0 * s_1 * \cdots * s_m$ in such a way that the lexicographic ordering on Y treats each s_i as a unit; so the lexicographic ordering on Y mimics the lexicographic ordering on $K(m)$.

Lemma 2.6. *Suppose κ is a regular cardinal, n and m are positive integers, X is a subset of $K(n)$, the n-fold cartesian product of κ with itself, and X has type at least κ^m under the lexicographic ordering. Then for some $(m+1)$ element set I, X has a subset Y which is an I-product set.*

Proof. The proof proceeds by induction on n. If $n = m = 1$, then X is a $\{0,1\}$-product set.

So assume that $n > 1$ and that the lemma holds for $n-1$. For each $a \in \kappa$, let $X(a)$ be the set of sequences in X which begin with a and let $T(a)$ be the set of tails of those sequences from X that begin with a. That is, t is in $T(a)$ if and only if $(a) * t$ is in $X(a)$.

Case 1. For some a, $T(a)$ has type at least κ^m.

Since $T(a) \subseteq K(n-1)$, by the induction hypothesis there are sets J and Z so that

(i) $Z \subseteq T_a$,

(ii) $J \subset n$ and $|J| = m+1$,

(iii) Z is a J-product set.

Set $I = \{j+1 : j \in J\}$ and $Y = \{(a) * z : z \in Z\}$. Then Y is an I-product set and $Y \subset X$.

Case 2. For all a, $T(a)$ has type less than k^m.

Let A be the set of all a so that $T(a)$ has type at least κ^{m-1}. Then $V = \bigcup\{X(b) : b \in \kappa - A\}$ has type at most κ^{m-1}. So $X - V = \bigcup\{X(a) : a \in A\}$ has type at least κ^m and A has cardinality κ.

For each $a \in A$, since $T(a) \subseteq K(n-1)$, by the induction hypothesis there are sets $J(a)$ and $Z(a)$ so that

(i) $Z(a) \subseteq T(a)$,

(ii) $J(a) \subseteq n$ and $|J(a)| = n$,

(iii) $Z(a)$ is a $J(a)$-product set.

Since there are only finitely many subsets of n and A is infinite, there are sets $J \subset n$ and $C \subset A$ so that $|C| = \kappa$ and for all $c \in C$, $J(c) = J$. For $c \in C$, let $Y(c) = \{(c) * z : z \in Z(c)\}$. Set $I = \{0\} \cup \{j+1 : j \in J\}$ and $Y = \bigcup\{Y(c) : c \in C\}$. Then Y is an I-product set and $Y \subset X$.

LEMMA 2.7. *There is no subset $X \subset K$ of type $\kappa^{\omega+1}$ with $R[X]^2 = \{0\}$.*

PROOF. Let $X \subset K$ be a set of type $\kappa^{\omega+1}$. To prove the lemma we must find $\{u, v\} \subset X$ with $R\{u, v\} = 1$.

For each a in κ, let $X(a)$ be the set of sequences in X that begin with a. Let A be the set of a so that $X(a)$ has type at least κ^3. Then A has type κ, since X has type $\kappa^{\omega+1}$. For each a in A, since κ is a regular uncountable cardinal, by Lemma 2.3, there is some n so that $X(a) \cap K(n)$ has type at least κ^3. Now on $X(a) \cap K(n)$ the ordering $<$ agrees with the lexicographic ordering. So for each a in A, there are n, $I(a)$ and $Y(a)$ so that $Y(a) \subset X(a) \cap K(n)$ and $Y(a)$ is an $I(a)$-product set.

Since A has order type κ and there are only countably many four element sets of natural numbers, there is a set $B \subset A$ of order type κ, and a set I so that for all b in B, $I(b) = I$. Let i, j, k and n be the elements of I listed in increasing order.

Let u_0 be in B and let $(u_0, u_1, \cdots, u_{j-1})$ be an element of $Y(u_0) \,|\, j$. Let v_0 be an element of B with $v_0 > u_{j-1}$. There must be an element of B greater than u_{j-1}, since B has order type κ. Let $(v_0, v_1, \cdots, v_{i-1})$ be an element of $Y(v_0) \,|\, i$. Since $Y(u_0)$ and $Y(v_0)$ are both I-product sets, there are sequences $(u_j, u_{j+1}, \cdots, u_{k-1})$,

$(v_i, v_{i+1}, \cdots, v_{j-1})$, $(u_k, u_{k+1}, \cdots, u_{n-1})$, $(v_j, v_{j+1}, \cdots, v_{n-1})$ so that $(u_0, u_1, \cdots, u_{n-1})$ is in $Y(u_0)$, $(v_0, v_1, \cdots, v_{n-1})$ is in $Y(v_0)$ and $u_j > v_{i-1}$, $v_i > u_{k-1}$, $u_k > v_{j-1}$, $v_j > u_{n-1}$. Then $u_{j-1} < v_0 \leqq u_j < v_{j-1} \leqq u_{n-1} < v_j$, so $R\{(u_0, u_1, \cdots, u_{n-1}), (v_0, \cdots, v_{n-1})\} = 1$.

§3. Pinning and consequences

Let A and B be well-ordered sets. A function $s : A \to B$ is called a pinning map in case, for every subset $X \subset A$ order-isomorphic to A, its image $s(X)$ is order-isomorphic to B. If α and β are ordinals, then α *can be pinned to* β, in symbols $\alpha \twoheadrightarrow \beta$, if there is a pinning map from α into β. Clearly if $\operatorname{tp} A = \alpha$ and $\operatorname{tp} B = \beta$, then $\alpha \twoheadrightarrow \beta$ if and only if there is a pinning map from A into B.

E. Specker introduced this notion in [14] to transfer results about one ordinal to another. The basic lemma relating partition relations to pinning is the following.

LEMMA 3.1. *Suppose α, β are ordinals and m a cardinal. If $\alpha \twoheadrightarrow \beta$ and $\beta \not\to (\beta, m)^2$, then $\alpha \not\to (\alpha, m)^2$.*

PROOF. Let $s : \alpha \twoheadrightarrow \beta$ be a pinning map. Suppose $r : [\beta]^2 \to \{0, 1\}$ is a counter-example to $\beta \to (\beta, m)^2$. Define $R : [\alpha]^2 \to \{0, 1\}$ by setting $R\{\xi, \eta\} = 1$ if and only if $s(\xi) \neq s(\eta)$ and $r\{s(\xi), s(\eta)\} = 1$. Then R is a counter-example to $\alpha \to (\alpha, m)^2$.

Obviously more can be proved, but this lemma suffices for the present purposes.

J. Baumgartner [1] used pinning to derive consequences of Theorem 2.1. He showed that the theorem could be extended to cardinals of uncountable cofinality. He also showed that if κ is a cardinal of uncountable cofinality and α is an ordinal with $\kappa^\omega < \alpha < \kappa^+$ and $\operatorname{cf} \alpha = \operatorname{cf} \kappa$, then $\alpha \not\to (\alpha, 3)^2$. More specifically, he proved the following:

THEOREM 3.2. (J. Baumgartner [1]) *Assume κ is a cardinal such that $\operatorname{cf} \kappa > \omega$. Let α be an indecomposable ordinal such that $\kappa^\omega < \alpha < \kappa^+$ and $\operatorname{cf} \alpha = \lambda$. Then $\alpha \twoheadrightarrow \kappa^\omega \cdot \lambda$. If $\operatorname{cf} \kappa = \mu$, then $\alpha \twoheadrightarrow \mu^\omega \cdot \lambda$ also.*

COROLLARY 3.3. (J. Baumgartner [1]) *Assume κ is a cardinal such that $\operatorname{cf} \kappa > \omega$. Let α be an indecomposable ordinal such that $\kappa^\omega < \alpha < \kappa^+$. If $\operatorname{cf} \alpha = \operatorname{cf} \kappa$, then $\alpha \not\to (\alpha, 3)^2$. In particular, $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ and $\kappa^\kappa \not\to (\kappa^\kappa, 3)^2$.*

Baumgartner's proof of Theorem 3.2 may be modified slightly to prove the following theorem:

THEOREM 3.4. *Assume κ is a cardinal with cofinality $\lambda > \omega$, and a is an ordinal with $\omega < a < \kappa^+$. If a is decomposable, then $\kappa^a \to \lambda^{\omega+1}$.*

The above Theorem 3.4 on pinning, the main Theorem 2.1 of the paper and Lemma 3.1 connecting pinning and partition relations combine to give the following corollary:

COROLLARY 3.5. *Assume that κ is a cardinal with uncountable cofinality and that a is an ordinal with $\omega < a < \kappa^+$. If a is decomposable, then $\kappa^a \not\to (\kappa^a, 3)^2$.*

The rest of this section is devoted to the proof of Theorem 3.4.

The lemmas Baumgartner used are the following:

LEMMA 3.5. (6.2 on page 305 [1]) *Suppose κ is a singular cardinal and $\mathrm{cf}\,\kappa = \lambda > \omega$. Then $\kappa^\omega \to \lambda^\omega$. In fact, there is a function $f : \kappa^\omega \to \lambda^\omega$ such that for all $n < \omega$, if $X \subseteq \kappa^\omega$ is of order type κ^n, then $f(X)$ has order type $\geqq \lambda^n$.*

LEMMA 3.6. (Milner–Rado Paradoxical Decomposition [12]) *Let α be an ordinal number and let κ be a cardinal such that $\kappa \leqq \alpha < \kappa^+$. Then there are sets A_n for each $n < \omega$ such that $\alpha = \bigcup\{A_n : n < \omega\}$ and each A_n has order type less than κ^ω.*

LEMMA 3.7. (6.3 on page 306 [1]) *Let κ be a cardinal such that* $\mathrm{cf}\,\kappa > \omega$, *and let $\kappa^\omega \leqq \alpha < \kappa^+$. Then there is $f_\alpha : \alpha \to \kappa^\omega$ such that for every $A \subseteq \alpha$ and every $n < \omega$, if A has order type κ^n, then $f(A)$ has order type $\geqq \kappa^n$. Therefore, if $A \subset \alpha$ has order type κ^ω, then so does $f(A)$. In particular, $\alpha \to \kappa^\omega$.*

PROOF OF THEOREM 3.4. Let κ be a cardinal with cofinality $\lambda > \omega$, and let a be a decomposable ordinal with $\omega < a < \kappa^+$. Then the theorem states that $\kappa^a \to \lambda^{\omega+1}$.

Since a is decomposable, $a = b + c$ where $b \geqq c$ and c is indecomposable. If $c = 1$, then $\mathrm{cf}\,\kappa^a = \mathrm{cf}\,\kappa$, and the theorem follows from Theorem 3.2. So assume $c > 1$. Let B be a set of type κ^b, C a set of type κ^c, and let $f : B \to \lambda^\omega$ be the composition of the pinning maps described in Lemmas 3.7 and 3.5. Since C has cardinality κ, there is a decomposition of $C = \bigcup_{\alpha<\lambda} C_\alpha$ into λ disjoint sets each of cardinality less than κ. Now $C \times B$ has order type $\kappa^b \cdot \kappa^c$ under the lexicographic ordering, and $\lambda \times \lambda^\omega$ has type $\lambda^\omega \cdot \lambda = \lambda^{\omega+1}$. Define $r : C \times B \to \lambda \times \lambda^{\omega+1}$ by $r((\gamma, \beta)) = (\alpha, f(\beta))$ where $\gamma \in C_\alpha$. It suffices to show r is the desired pinning map.

Suppose $X \subset C \times B$ and type $X = \kappa^a$. For each $\alpha < \lambda$, let $Y_\alpha = \{\beta \in B : \exists\gamma(\gamma, \beta) \in X \cap (C_\alpha \times B)\}$.

CLAIM. *For every $n < \omega$ and every $\alpha' < \lambda$, there is $\alpha > \alpha'$ so that* $\operatorname{tp} Y_\alpha \geqq \kappa^n$.

PROOF OF CLAIM. Assume by way of contradiction that there are n and α' so that for all $\alpha > \alpha'$, $\operatorname{tp} Y_\alpha < \kappa^n$. Let $D = \bigcup\{C_\alpha : \alpha \leqq \alpha'\}$. Since $\alpha' > \lambda$, where λ is the cofinality of κ and each C_α has cardinality less than κ, it follows that $\operatorname{tp} D = \mu < \kappa$. So $U = X \cap (D \times B) = \bigcup\{X \cap (C_\alpha \times B) : \alpha \leqq \alpha'\}$ has type at most $\kappa^b \cdot \mu < \kappa^b \cdot \kappa < \kappa^a$. Let

$$V = X \cap ((C - D) \times B) = \bigcup\{X \cap (C_\alpha \times B) : \alpha' < \alpha < \lambda\}.$$

Then $X = U \cup V$. Since $\operatorname{tp} X = \kappa^a$ is indecomposable, and $\operatorname{tp} U < \kappa^a$, it follows that $\operatorname{tp} V = \kappa^a$. However, if $\alpha' < \alpha$ and $s \in C_\alpha$, then $X \cap (\{s\} \times B) \subset X \cap (\{s\} \times Y_\alpha)$, so $\operatorname{tp} X \cap (\{s\} \times B) < \kappa^n$. Thus $\operatorname{tp} V \leqq \kappa^n \cdot \kappa < \kappa^a$. Therefore $\kappa^a = \operatorname{tp} V < \kappa^a$. This contradiction yields the claim.

Since f is the composition of the pinning maps described in Lemmas 3.7 and 3.5, if Y_α has type at least κ^n, then $f(Y_\alpha)$ has type at least λ^n. So if Y_α has type at least κ^n, then $r(X \cap (C_\alpha \times B))$ is a subset of $\{\alpha\} \times \lambda^\omega$ of type at least λ^n. This implication and the claim above yield the conclusion that for every $n < \omega$ and every $\alpha' < \lambda$, there is an $\alpha > \alpha'$ and a subset of $r(X) \cap (\{\alpha\} \times \lambda^{\omega+1})$ of type at least λ^n. But the conclusion suffices to show that $r(X)$ is a set of type $\lambda^{\omega+1}$. So r is the desired pinning map.

§4. Limitations

In this section, I characterize for a regular uncountable cardinal κ the set of ordinals of cardinality κ that can be pinned to $\kappa^{\omega+1}$. The first step is to characterize the ordinals of power κ that can be pinned to κ^2. The second step is to use the results of the previous section to show that if $\alpha \geqq \kappa^{\omega+1}$ and α can be pinned to κ^2, then α can be pinned to $\kappa^{\omega+1}$. Now $\kappa^{\omega+1}$ can be pinned to κ^2, and pinning is transitive. So if $\alpha \geqq \kappa^{\omega+1}$, then α can be pinned to $\kappa^{\omega+1}$ if and only if α can be pinned to κ^2.

If κ is a singular cardinal of uncountable cofinality λ, then one is interested in the set of ordinals which can be pinned to $\lambda^{\omega+1}$. The procedure used to determine the set for a regular cardinal is stymied at the first step since extra set theoretic assumptions are required to determine the pinning relation $\alpha \to \lambda^2$ for some α. A similar situation occurs for cardinals of cofinality ω, although in that case, since if κ has cofinality ω, then $\kappa^{\omega+1}$ can be pinned to ω^3, the set of interest is the set of ordinals which can be pinned to ω^3. The difficulties that arise are exemplified by the following result.

THEOREM 4.1. (See [11]) *If the Continuum Hypothesis is assumed, then* $\omega_1^{\omega+2} \to \omega^2$. *On the other hand, if Martin's Axiom holds and the Continuum Hypothesis fails, then* $\omega_1^{\omega+2} \not\to \omega^2$.

The next two lemmas concern ordinals which cannot be pinned to κ^2.

LEMMA 4.2. *If κ is a regular cardinal and a is an indecomposable ordinal with* $0 < a < \kappa^+$ *and* $\operatorname{cf} a \neq \kappa$, *then* $\kappa^a \not\to \kappa^2$.

PROOF. Let $\mu = \operatorname{cf} a$. Since a is indecomposable, a is the limit of an increasing sequence $\langle a(\nu) : \nu < \mu \rangle$. And a is also the limit of the increasing sequence $\langle a(\nu) \cdot 2 : \nu < \mu \rangle$. So κ^a is the limit of $\langle \kappa^{a(\nu)} : \nu < \mu \rangle$ and of $\langle \kappa^{a(\nu) \cdot 2} : \nu < \mu \rangle$. Let $M(\nu) = \{\nu\} \times \kappa^{a(\nu)} \times \kappa^{a(\nu)}$, and let $M = \bigcup \{M(\nu) : \nu < \mu\}$. Under the lexicographic ordering M has type κ^a.

To prove the lemma it suffices to show there is no pinning map from M to κ^2. So suppose $f : M \to \kappa^2$ is given. For each $\nu < \mu$, look at f restricted to $M(\nu)$. If for some s in $\kappa^{a(\nu)}$, the image of f restricted to $\{\nu\} \times \{s\} \times \kappa^{a(\nu)}$ is bounded, call ν good, and set $X(\nu) = \{\nu\} \times \{s\} \times \kappa^{a(\nu)}$ for the smallest s which has this property, and let $b(\nu)$ be a bound for the image of $\{\nu\} \times \{s\} \times \kappa^{a(\nu)}$.

Let G be the set of all good ν. First suppose G has cardinality μ. Since $\mu < \kappa$ $b(G)$ has cardinality less than κ. Let $X = \bigcup \{X(\nu) : \nu \in G\}$. Then X has type κ^a and f is bounded on X. So f cannot be a pinning map if G has cardinality μ.

So suppose G does not have cardinality μ. Then $H = \mu - G$ has cardinality μ. Enumerate $Y = \{\{\nu\} \times \kappa^{a(\nu)} : \nu \in H\}$ in order type κ as $\{(x_\gamma, y_\gamma) : \gamma < \kappa\}$. Now use the fact that for $\nu = x_\gamma$ in H, f is unbounded on $\{x_\gamma\} \times \{y_\gamma\} \times \kappa^{a(\nu)}$. By recursion define $\{z_\gamma : \gamma < \kappa\}$ so that if $\gamma > \delta$, then $f(x_\gamma, y_\gamma, z_\gamma) > f(x_\delta, y_\delta, z_\delta)$. Let $X = \{(x_\gamma, y_\gamma, z_\gamma) : \gamma < \kappa\}$. In lexicographic ordering, X and Y have the same order type, and that common order type is κ^a. Moreover, the image $f(X)$ has order type κ. So f cannot be a pinning map.

LEMMA 4.3. *Assume κ is a regular cardinal, γ and β are indecomposable ordinals with* $\beta < \kappa$ *and* $\gamma < \kappa^+$. *If* $\gamma \not\to \kappa^2$, *then* $\gamma \cdot \beta \not\to \kappa^2$.

PROOF. The set $\beta \times \gamma$ ordered lexicographically has order type $\gamma \cdot \beta$. Suppose a function $f : \beta \times \gamma \to \kappa^2$ maps $\beta \times \gamma$ into κ^2. For each $b < \gamma$, use the hypothesis that $\gamma \not\to \kappa^2$ to find a subset $X(b) \subset \{b\} \times \gamma$ of type γ so that the image has type less than κ^2. Let $X = \bigcup \{X(b) : b < \beta\}$. Then X has type $\gamma \cdot \beta$. The image $f(X)$ is the union of $f(X(b))$ for $b < \beta$, and each $f(X(b))$ has type less than κ^2. Since κ is regular, by Lemma 2.3 $\kappa^2 \to (\kappa^2)^1_\nu$ where $\nu = |\beta| < \kappa$. So κ^2 is not the union of β sets of type less than κ^2. Thus $f(X)$ has type less than κ^2, and f is not a pinning map. Therefore $\gamma \cdot \beta \not\to \kappa^2$.

The next theorem relates pinning to κ^2 to pinning to κ^3 and $\kappa^{\omega+1}$ for a regular uncountable cardinal κ, and it will be used in the general discussion of the relation $\alpha \to (\alpha, 3)^2$ that appears in the next section.

THEOREM 4.4. *Assume κ is a regular cardinal and α is an indecomposable ordinal with $\kappa^3 \leqq \alpha < \kappa^+$. Then the following are equivalent*:

(1) $\alpha \to \kappa^2$.

(2) $\alpha = \kappa^a \cdot \beta$ *where $\beta < \kappa^+$ and either* cf $a = \kappa$ *or a is decomposable.*

(3) *Either* $(\alpha < \kappa^{\omega+1}$ *and* $\alpha \to \kappa^3)$ *or* $(\alpha \leqq \kappa^{\omega+1}$ *and* $\alpha \to \kappa^{\omega+1})$.

(4) *Either* $\alpha \to \kappa^3$ *or* $\alpha \to \kappa^{\omega+1}$.

It is not hard to show that if $\alpha < \kappa^{\omega+1}$, then α cannot be pinned to $\kappa^{\omega+1}$. So the theorem yields a characterization of the set of ordinals which can be pinned to $\kappa^{\omega+1}$.

PROOF. Let α be an indecomposable ordinal with $\kappa^3 \leqq \alpha < \kappa^+$. Then by work of W. Sierpinski (see [13] theorem 3 on page 325), there are a and β so $\alpha = \kappa^a \cdot \beta$ where $0 \leqq \alpha < \kappa^+$ and β is an indecomposable ordinal with $0 < \beta < \kappa$. Since $\alpha \geqq \kappa^3$, it follows that $a \geqq 3$.

Now by the previous two lemmas, (1) implies (2).

Pinning is transitive. And $\kappa^a \cdot \beta$ can be pinned to κ^a. If $3 \geqq a < \omega$, then κ^a can be pinned to κ^3. So if $3 \leqq a < \omega$, then $\alpha = \kappa^a \cdot \beta$ can be pinned to κ^3. If $\omega \leqq a < \kappa^+$ and either a is decomposable or cf $a = \kappa$, then κ^a can be pinned to $\kappa^{\omega+1}$ by Theorems 3.4 and 3.2. So by the transitivity of pinning, if $\omega \leqq a < \kappa^+$, then $\alpha = \kappa^a \cdot \beta$ can be pinned to $\kappa^{\omega+1}$. Thus (2) implies (3).

Trivially, (3) implies (4). Since both κ^3 and $\kappa^{\omega+1}$ can be pinned to κ^2 and since pinning is transitive, the implication (4) implies (1) holds. Thus the four conditions are equivalent.

§5. A general discussion

This section is a discussion of the relation $\alpha \to (\alpha, 3)^2$ for ordinals α of the form $\alpha = \kappa^\beta$ where κ is a cardinal and $0 < \beta < \kappa^+$. It is based both on the counter-example of the paper and on results of other mathematicians. The pinning relation, which was introduced by E. Specker, plays an important role in the discussion. Its first appearance in this paper is in section 3 where it is defined and where the basic lemma is proved which says that if α can be pinned to β (in symbols, we write $\alpha \to \beta$), and $\beta \not\to (\beta, 3)^2$, then $\alpha \not\to (\alpha, 3)^2$.

It is not hard to see that if a non-zero ordinal α is decomposable, then $\alpha \not\to (\alpha, 3)^2$. And indecomposable ordinals can be written as the product of ordinal powers of cardinals. For if α and μ are ordinals, then α can be written uniquely as $\alpha = \mu^{a_1}\cdot\nu_1 + \nu^{a_2}\cdot\nu_2 + \cdots + \mu^{a_{n-1}}\cdot\nu_{n-1}$, where $a_1 \geqq a_2 \geqq \cdots \geqq a_{n-1}$ and $\nu_1 < \mu$, $\nu_2 < \mu, \cdots, \nu_{n-1} < \mu$ (see p. 325 of Sierpinski's book [13]). So if α is indecomposable and $\kappa = |\alpha|$, then $\alpha = \kappa^\beta \cdot \nu$ where $0 < \beta < \kappa^+$ and $\nu < \kappa$. By induction it follows that every indecomposable ordinal α is the product $\alpha = \alpha_0 \cdot \alpha_1 \cdot \cdots \cdot \alpha_{n-1}$ of ordinal powers of cardinals $\alpha_i = \kappa_i^{\beta(i)}$ where further $\kappa_0 > \kappa_1 > \cdots > \kappa_{n-1}$. Such a product can be represented in order type as a Cartesian product with a lexicographic ordering. With such a representation, the projection maps show that α can be pinned to each factor, $\alpha \to \alpha_i$. So if some $\alpha_i = \kappa_i^{\beta(i)}$ satisfies $\alpha_i \not\to (\alpha_i, 3)^2$, then α satisfies $\alpha \not\to (\alpha, 3)^2$. So ordinals of the form κ^β are basic in the study of the relation $\alpha \to (\alpha, 3)^2$.

Only three infinite countable ordinals are known to have the positive relation $\alpha \to (\alpha, 3)^2$. And all the known negative relations for indecomposable ordinals depend on the negative result for one ordinal, ω^3. The partition relation $\omega \to (\omega, \omega)^2$ is a special case of Ramsey's Theorem. E. Specker [14] proved $\omega^2 \to (\omega^2, m)^2$ for all $m < \omega$ and proved $\omega^3 \not\to (\omega^3, 3)^2$, the basic counter-example for countable ordinals. He introduced pinning, showed $\omega^n \to \omega^3$ for all n with $3 \leqq n < \omega$, and concluded $\omega^n \not\to (\omega^n, 3)^2$. C. C. Chang [2] proved $\omega^\omega \to (\omega^\omega, 3)^2$, and E. C. Milner improved his result to $\omega^\omega \to (\omega^\omega, m)^2$ for all $m < \omega$. For a shorter proof of that result, see [9]. F. Galvin [5] proved that if $a \geqq 3$ is countable and decomposable, then ω^a can be pinned to ω^3, so $\omega^a \not\to (\omega^a, 3)^2$. I showed that the counter-example $\omega^3 \not\to (\omega^3, 3)^2$ cannot be extended through pinning to other countable ordinals by showing that if a countable indecomposable ordinal $\alpha = \omega^a$ can be pinned to ω^3, then a must be decomposable (see [5]). For countable ordinals of the form $\alpha = \omega^a$ where $a > \omega$ is indecomposable, nothing is known about the relation $\alpha \to (\alpha, 3)^2$. In particular $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$ and $\omega^{\omega^\omega} \to (\omega^{\omega^\omega}, 3)^2$ are open.

The results for countable ordinals can be generalized, sometimes in restricted cases, to uncountable ordinals. If κ is an uncountable cardinal, then it is well-known that $\kappa \to (\kappa, \omega)^2$ (see [4]). Specker's result for ω^2 can be generalized to $\kappa^2 \to (\kappa^2, 3)^2$ if κ is a weakly compact cardinal; an uncountable cardinal is weakly compact if $\kappa \to (\kappa)^2_2$. J. Baumgartner [1] has shown that if κ is a strong limit cardinal and $m < \omega$, then $\kappa^2 \to (\kappa^2, m)^2$ if and only if $(\mathrm{cf}\,\kappa)^2 \to ((\mathrm{cf}\,\kappa)^2, m)^2$. In the same paper he also proved that if κ is regular and $\kappa^2 \to (\kappa^2, 3)^2$, then the κ-Souslin hypothesis holds. He uses Jensen's result [7] that if $V = L$, then κ is weakly compact if and only if the κ-Souslin hypothsis holds to derive the following corollary: If $V = L$ and κ is a cardinal, then $\kappa^2 \to (\kappa^2, 3)^2$ if and only if $\mathrm{cf}\,\kappa$ is weakly compact. In [6], A. Hajnal proved that if κ is regular and $2^\kappa = \kappa^+$, then $(\kappa^+)^2 \not\to ((\kappa^+)^2, 3)^2$. J. Baumgartner [1] generalized Hajnal's result by proving that for both regular and singular cardinals, if $2^\kappa = \kappa^+$, then $(\kappa^+)^2 \not\to ((\kappa^+)^2, 3)^2$.

Specker's argument for ω^3 can be generalized to show that if κ is a regular cardinal, then $\kappa^3 \not\to (\kappa^3, 3)^2$. If κ is a cardinal, and n is an ordinal with $3 \leqq n < \omega$, then $\operatorname{cf}\kappa$ is regular and κ^n can be pinned to $(\operatorname{cf}\kappa)^3$, so $\kappa^n \not\to (\kappa^n, 3)^2$.

If κ is a Ramsey cardinal, then the Chang–Milner result for ω^ω can be generalized to prove $\kappa^\omega \to (\kappa^\omega, m)^2$ for all $m < \omega$ (see [8] for a proof; a cardinal κ is Ramsey if $\kappa \to (\kappa)_2^{<\omega}$). This positive result using a Ramsey cardinal shows that there can be no simple combinatorial counter-example like the one used to show $\omega^3 \not\to (\omega^3, 3)^2$ for ordinals of the form κ^ω where κ is an uncountable cardinal. In [3], A. Hajnal proved that if $2^\kappa = \kappa^+$, then $\kappa^+ \cdot \kappa \not\to (\kappa^+ \cdot \kappa, 3)^2$. Now ω_1^ω can be pinned to $\omega_1 \cdot \omega$. So if the continuum hypothesis holds, Hajnal's result yields $\omega_1^\omega \not\to (\omega_1^\omega, 3)^2$. In contrast with Hajnal's negative result, $\kappa^+ \cdot \kappa \not\to (\kappa^+ \cdot \kappa, 3)^2$, J. Baumgartner [1] has shown that if κ is a weakly compact cardinal and $\alpha < \kappa$ is an ordinal with $\alpha \to (\alpha, m)^2$, then $\kappa \cdot \alpha \to (\kappa \cdot \alpha, 3)^2$. I generalized his result to show under the same hypotheses that $\kappa^2 \cdot \alpha \to (\kappa^2 \cdot \alpha, 3)^2$. For proofs, see [10].

The main theorem of this paper is that if κ is a regular cardinal, then $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$. If κ is a singular cardinal of uncountable cofinality, then $\kappa^{\omega+1}$ can be pinned to $(\operatorname{cf}\kappa)^{\omega+1}$. If κ has cofinality ω, then $\kappa^{\omega+1}$ can be pinned to ω^3. So for every cardinal κ, the negative relation $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ holds. J. Baumgartner [1] proved that if κ is a cardinal of uncountable cofinality and α is an ordinal with $\kappa^\omega < \alpha < \kappa^+$ and with $\operatorname{cf}\alpha = \operatorname{cf}\kappa > \omega$, then α can be pinned to $(\operatorname{cf}\kappa)^{\omega+1}$, so $\alpha \not\to (\alpha, 3)^2$. Using his methods, I proved that if κ is a cardinal of uncountable cofinality, a is a decomposable ordinal with $\omega < a < \kappa^+$, then $\alpha = \kappa^a$ can be pinned to $(\operatorname{cf}\kappa)^{\omega+1}$, so $\alpha \not\to (\alpha, 3)^2$ (see section 3). If κ is a regular uncountable cardinal, then these are the only kinds of ordinals that can be pinned to $\kappa^{\omega+1} = (\operatorname{cf}\kappa)^{\omega+1}$ (see section 4).

Now $\kappa^{\omega+1}$ can be pinned to κ^2 and to $(\operatorname{cf}\kappa)^2$. So if κ is the cardinal successor of λ and $2^\lambda = \kappa$, then the theorem $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ is a corollary to the results of Hajnal and Baumgartner. Similarly, if $V = L$ and $\operatorname{cf}\kappa$ is not weakly compact, then the relation $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ follows from $(\operatorname{cf}\kappa)^2 \not\to ((\operatorname{cf}\kappa)^2, 3)^2$. However, if $\kappa^2 \to (\kappa^2, 3)^2$ as it does for a weakly compact cardinal κ, then the results $\kappa^3 \not\to (\kappa^3, 3)^2$ and $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ give all the known negative results. Furthermore, if κ is regular, then for ordinals $\alpha \geqq \kappa^3$, the two results $\kappa^3 \not\to (\kappa^3, 3)^2$ and $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$ are as powerful as $\kappa^2 \not\to (\kappa^2, 3)^2$, since if $\alpha \geqq \kappa^3$, then α can be pinned to κ^2 if and only if it can be pinned to one of κ^3 and $\kappa^{\omega+1}$. (See section 4.)

If κ and λ are both Ramsey cardinals and $\kappa > \lambda$, then I have shown (unpublished) that $\kappa^{\lambda} \to (\kappa^{\lambda}, 3)^2$. One of Baumgartner's results shows that for no uncountable cardinal κ does $\kappa^{\kappa} \to (\kappa^{\kappa}, 3)^2$.

The following questions are open.

(1) $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$?

(2) $\omega^{\omega^{\omega}} \to (\omega^{\omega^{\omega}}, 3)^2$?

(3) $\omega_2 \cdot \omega \to (\omega_2 \cdot \omega, 3)^2$?

(4) $\omega_2^{\omega} \to (\omega_2^{\omega}, 3)^2$?

(5) Is there a cardinal κ of uncountable cofinality and an ordinal α with $\kappa^{\kappa} < \alpha < \kappa^{+}$ so that $\alpha \to (\alpha, 3)^2$?

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